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ON THE UNIQUENESS OF SOLUTION AND CERTAIN APPROXIMATE METHODS OF SOLVING LINEAR VISCOELASTICITY PROBLEMS

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The uniqueness of solution of mixed initial-boundary value problems for a version of linear viscoelasticity is proved, as is also the convergence of some approximate methods of solving these problems. Upon compliance with the conditions $\mu > 0$, $m_A \leqslant m_B$ and if $m_A = m_B$, then $\lambda + 2\mu > 0$ the following cases of the relationship between powers of the polynomials B and C are considered: $m_B \geqslant m_C + 2$ (Sect. 3), $m_B = m_C$ and $m_B = m_C + 1$ (Sect. 4). These cases include all the types of relations between the strains and stresses of the type (1.1) used in the theory of viscoelasticity. As a mathematical foundation for such constraints, it can be noted that for $m_C > m_B$ the Cauchy problem for a body occupying all of space is always incorrect according to G. E. Shilov [1].

1. A version of linear viscoelasticity is considered with the following governing relationships between the stress σ_{ij} and strain ε_{ij} tensors:

$$C(\partial_t)\,\sigma_{ij} = \lambda A(\partial_t)\,\theta\delta_{ij} + 2\mu B(\partial_t)\,\varepsilon_{ij} \tag{1.1}$$
$$\varepsilon_{ij} = {}^1/_2\,(u_{i,j} + u_{j,i})$$

Here $A(p)$, $B(p)$ and $C(p)$ are some polynomials of the variable p with the highest coefficient equal to unity and the powers m_A, $m_B = n$, $m_C = m - 2$, respectively, λ and μ are the instantaneous elastic moduli, $\theta = \varepsilon_{ii}$ is the volume strain, $u_{i,j}$ is the partial derivative of the ith component of the displacement vector $\mathbf{u}$ of points of the body with respect to the variable x_j, ∂_t is the partial derivative with respect to time t, and δ_{ij} is the Kronecker symbol; summation is performed over the repeated subscripts.

It is natural to use the principle of possible displacements in the Lagrange form

$$\int_\Omega \sigma_{ij}\delta\varepsilon_{ij}\,d\Omega = \int_\Omega (F_i - \rho\partial_t^2 u_i)\,\delta u_i\,d\Omega + \int_S P_i\delta u_i\,dS \tag{1.2}$$

for a generalized formulation of viscoelasticity theory problems, where δu_i is the variation of the displacement vector components, $\mathbf{F} = (F_1, F_2, F_3)$ are the volume forces, $\mathbf{P} = (P_1, P_2, P_3)$ are forces acting on the boundary S of the volume Ω occupied by the body, and ρ is the body density. Considering the variations δu_i independent of time t, let us eliminate σ_{ij} from (1.2) by applying the operator $C(\partial_t)$ to (1.2) and taking account of (1.1). If the vector function $\mathbf{u}$ in the equality thus obtained is a vector of the actual displacements of the body points, then this equality must be satisfied at any time t and for all arbitrary possible displacements δu_i, and it hence continues to be satisfied if δu_i depend on time t. Let us present this equality which we have additionally integrated with respect to time

$$\int_0^T \int_\Omega \{\lambda A(\partial_t)\theta\delta_{ij} + 2\mu B(\partial_t)\varepsilon_{ij}\}\delta\varepsilon_{ij}\,d\Omega\,dt =$$

$$\int_0^T \int_\Omega \delta u_i C(\partial_t)(F_i - \rho\partial_t^2 u_i)\,d\Omega\,dt + \int_0^T \int_S \delta u_i C(\partial_t)P_i\,dS\,dt \tag{1.3}$$

The equality (1.3) is taken as the basis in the generalized formulation of linear visco-elasticity problems below.

For a complete formulation of the problem it is necessary to give the following initial and boundary conditions:

$$\partial_t^k \mathbf{u}\big|_{t=0} = \mathbf{a}_k, \quad k = 0, \ldots, \max(n-1, m-1) \tag{1.4}$$

$$N_1\mathbf{u}\big|_S = N_1\psi \tag{1.5}$$

$$N_2\sigma_{kl}\nu_l\mathbf{i}_k\big|_S = N_2\mathbf{P} = \mathbf{P} \tag{1.6}$$

where ν_l are direction cosines of the exterior normal to the surface S, and N_1, N_2 are mutually orthogonal projection operators in the space E^3 which depend piecewise-continuously on the coordinates of the surface S. The assignment of displacements or stresses on the body boundary, respectively, denotes the case of identical operators N_1 or N_2

It is furthermore considered that there is a degenerate part S_1 of the boundary S where the operator N_1 is identical, i.e. the possibility of body displacement as a rigid whole is eliminated, and the operator $N_1 + N_2$ is identical for all points of S.

The following condition must be satisfied.

Consistency condition. There exists a vector function Φ satisfying the initial and boundary conditions (1.4), (1.5).

2. Let us introduce certain spaces and let us examine their properties. The spaces $C^k[0, T; H]$, $L_p[0, T; H]$, $W_p^k(\Omega)$ are considered known [2]. That the vector function belongs to any of them means that each of its components belongs to this space.

The boundary S of the bounded volume Ω possesses a normal varying piecewise-continuously on S and such that some cone of positive aperture can be adjoined from within at each point S.

Definition 2.1. The space $B_1^{n,m}(T)$ is the closure of the set of vector-functions $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{a} \in C^\infty(\Omega \times [0, T])$ in the norm

$$\|\mathbf{a}\|^2_{B_1^{n,m}(T)} = \int_0^T \left\{\sum_{k=0}^{n} \|\partial_t^k \mathbf{a}\|^2_{W_2^1(\Omega)} + \sum_{k=0}^{m} \|\partial_t^k \mathbf{a}\|^2_{L_2(\Omega)}\right\} dt \tag{2.1}$$

Let us introduce the following notation:

$$(\mathbf{a}\cdot\mathbf{b})_{H_1} = \int_\Omega \{\lambda\theta(\mathbf{a})\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{a})\}\varepsilon_{ij}(\mathbf{b})\,d\Omega \tag{2.2}$$

Definition 2.2. The closure of a subset of vector functions $\mathbf{a} \in C^1(\Omega)$ satisfying the homogeneous boundary conditions (1.5) is the space H_1 in the norm (2.2).

Definition 2.3. The closure of the subset of vector functions $\mathbf{a} \in C^\infty(\Omega \times [0, T])$ satisfying the homogeneous boundary and initial conditions (1.4), (1.5) in the norm

$$\|\mathbf{a}\|^2_{B_2^n(T)} = \int_0^T (\partial_t^n\mathbf{a}\cdot\partial_t^n\mathbf{a})_{H_1}\,dt \tag{2.3}$$

is called the space $B_2^n(T)$ if $\mu > 0, \lambda + 2\mu > 0$. The space $B_2^\circ(T)$ has the special notation $H_2(T)$.

Note. When $S_1 = S$, the condition $\lambda + 2\mu > 0$ can be replaced by the weaker condition $3\lambda + 2\mu > 0$.

Definition 2.4. The closure of the subset of vector-functions $\mathbf{a} \in C^\infty(\Omega \times [0, T])$ satisfying the homogeneous boundary conditions (1.5) in the norm

$$\|\mathbf{a}\|^2_{H_3(T)} = \int_0^T \{(\mathbf{a}\cdot\mathbf{a})_{H_1} + \|\partial_t \mathbf{a}\|^2_{L_2(\Omega)}\}\, dt \tag{2.4}$$

is the space $H_3(T)$. The subset $H_3(T)$ obtained by the closure of vector functions which equal zero in some neighborhood of $t = T$ for each function is denoted by $H_3^\circ(T)$.

Definition 2.5. $B_3^n(T)$ is the subset of vector functions $\mathbf{a} \in B_1^{n,n}(T)$ such that $\|\partial_t \mathbf{a}\|_{L_2(\Omega)}$ is a function bounded in $[0, T]$.

Definition 2.6. The space of elements $\mathbf{b}$ dual to the space H and with the norm

$$\|\mathbf{b}\|_{H^-} = \sup |(\mathbf{b}\cdot\mathbf{a})_{L_2(\Omega)}|\, \|\mathbf{a}\|_H^{-1}, \quad \mathbf{a} \in H$$

is called the negative space H^-.

Lemma 2.1. The following imbeddings of spaces with continuous imbedding operator hold:

$$H_1 \subset W_2^1(\Omega), \qquad L_{6/5}(\Omega) \subset H_1^-$$

$$B_2^n(T) \subset B_1^{n,n}(T), \quad B_1^{n,m}(T) \subset C^{n-1}[0, T; W_2^1(\Omega)]$$

$$B_1^{n,m}(T) \subset C^k[0, T; W_2^{1-(k+0.5-n)(m-n)^{-1}}(\Omega)] \quad \text{for } n \leqslant k < m$$

$$B_1^{n,m}(T) \subset W_2^n[0, T; W_2^{1/2}(\Gamma)], \quad H_3(T) \subset W_2^1(\Omega \times [0, T])$$

where the norm (2.2) and the norm $W_2^1(\Omega)$ are equivalent in the space H_1, and the norms (2.3) and $B_1^{n,n}(T)$ in $B_2^n(T)$; $\Gamma \in \Lambda_1$. Here Λ_1 is the class of Liapunov surfaces Γ, i.e. a finite number of domains Γ_k each of which is representable parametrically as $z_k = f_k(x_k, y_k)$, where f_k are continuously differentiable functions can be isolated on the surface Γ, where any interior part of the surface Γ belongs to the union of some strictly interior closed subdomains Γ_k for each such part. The proof results from Theorem 3.2 (see [2], Chapt. 1) and the Korn inequality.

Lemma 2.2. Let there be a set of m vector functions $\mathbf{a}_0, \ldots, \mathbf{a}_m$ with the following properties:

$$\mathbf{a}_k \in W_2^1(\Omega), \qquad k = 0, \ldots, n-1$$

$$\mathbf{a}_k \in W_2^{1-(k+0.5-n)(m-n)^{-1}}(\Omega), \quad k = n, \ldots, m-1, \qquad \text{if} \quad m > n$$

In this case there exists a vector function $\mathbf{a} \in B_1^{n,m}(T)$ taking this set as the initial values (1.4). If the functions $\mathbf{a}_k$ satisfy the homogeneous boundary conditions (1.5) in the sense of the spaces to which they belong, then the vector function $\mathbf{a}$ also satisfies the homogeneous boundary conditions (1.5). The proof results from Theorem 3.2 (see [2], Chapt. 1).

Lemma 2.3. For any vector function ω given on some surface $\Gamma \subset \Omega$, $\Gamma \in \Lambda_1$, such that $\partial_t^k \omega \in L_2[0, T; W_2^{1/2}(\Gamma)]$ $k = 0, \ldots, n-1$, there exists a vector function $\mathbf{a} \in B_1^{n,n}(T)$ taking the value ω on Γ, where if $\partial_t^k \omega = 0$, $k = 0, \ldots, n-1$ for $t = 0$, then $\partial_t^k \mathbf{a} = 0$ also for $t = 0$ and the reconstruction operator is continuous. The proof is carried out analogously to [3].

Lemmas 2.2 and 2.3 indicate important particular cases of the existence of the vector function $\mathbf{\Phi}$ from the consistency condition for a definite class. It must be noted that the representation

$$\mathbf{a}(x_i, t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} \partial_\tau{}^n \mathbf{a}(x_i, \tau)\, d\tau \tag{2.5}$$

is valid for a vector function $\mathbf{a} \in B_1^{n,m}(T)$ with initial values $\partial_t{}^k \mathbf{a} = 0$, $k = 0,..,n-1$ for $t = 0$.

3. Each of the three cases of the relationship between the powers of the polynomials B and C mentioned in Sect. 1 in connection with the singularities in the formulation of the problem, should be examined separately.

Definition 3.1. The vector function $\mathbf{u} \in B_1^{n,n}(T)$ satisfying the initial and boundary conditions (1.4), (1.5) such that the equality (1.3) is valid for any vector function $\delta\mathbf{u} \in H_2(T)$ is called the generalized solution of the linear viscoelasticity problem in the case $n = m_B \geqslant m_C + 2$.

The case $m_A = m_B$ is examined for definiteness. Let the vector function $\mathbf{\Phi}$ from the consistency condition be such that $\mathbf{\Phi} \in B_1^{n,n}(T)$. We seek the generalized solution of the linear viscoelasticity problem in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{\Phi} \tag{3.1}$$

where the vector function $\mathbf{u}_0$ evidently belongs to the space $B_2^n(T)$. By introducing the notation $\mathbf{v} = \partial_t{}^n \mathbf{u}_0$ and taking account of (2.5), (3.1), Eq. (1.3) can be reduced to the following form:

$$(\mathbf{v}\cdot\delta\mathbf{u})_{H_2(T)} = \int_0^T \{R_1(\mathbf{v}, \delta\mathbf{u}) + R_2(\mathbf{\Phi}, \mathbf{F}, \mathbf{P}, \delta\mathbf{u})\}\, dt \tag{3.2}$$

where R_k is some integro-differential form which is linear in each of its variables. It must be noted that the form $R_1(\mathbf{v}, \delta\mathbf{u})$ contains the variable $\mathbf{v}$ only under the integral with respect to time t, with the exception of the term $\rho\ (\mathbf{v}\cdot\delta\mathbf{u})_{L_2(\Omega)}$ for $m_C + 2 = m_B$, which must be added to the expression for the norm H_1 in this case. For $\rho > 0$ such a new norm H_1 is equivalent to the old one.

It can be shown that the uniqueness of solution of (3.2) in the space $H_2(T)$ is equivalent to the existence of a single generalized solution of the linear viscoelasticity problem. All the properties and methods of the solution of (3.2) are easily carried over directly to (1.3). Therefore, (3.2) will be investigated below, and the results again formulated for (1.3). The investigation is carried out analogously in Sect. 4.

Theorem 3.1. Let the following conditions be satisfied:

1) $m_C + 2 \leqslant m_B = n$, $m_A \leqslant m_B$, 2) $\mu > 0$, 3) if $m_A = m_B$, then $\lambda + 2\mu > 0$, 4) if $m_B = m_C + 2$, then $\rho > 0$, 5) $\mathbf{\Phi} \in B_1^{n,n}(T)$, 6) $C(\partial_t)\mathbf{F} \in L_2[0, T; H_1^{-}]$, 7) $C(\partial_t) P_i \in L_2[0, T; W_2^{-1/2}(S)]$

In this case: 1) there exists a single generalized solution of the linear viscoelasticity problem in the sense that has been introduced by Definition 3.1 ; 2) the solution can be sought approximately by the Bubnov-Galerkin method in the form $\mathbf{u}_n{}^* = c_1(t)\,\psi_1 + \dots + c_n(t)\,\psi_n$, where ψ_k is an orthonormal basis in H_1 and the system of equations of the Bubnov-Galerkin method is solvable at each step and the sequence $\mathbf{u}_n{}^*$ converges strongly to the solution of the problem in the space $B_1^{n,n}(T)$.

By estimating the members in the right side of (3.2) by using the Hölder inequality, it is easy to see that each of these members is a linear continuous functional in the variable $\delta\mathbf{u} = \varphi$ in the space $H_2(T)$. By the Riesz theorem on the representation of a continuous linear functional in Hilbert space, (3.2) can be written in the equivalent operator form

$$(\mathbf{v}\cdot\varphi)_{H_2(T)} = (G_T\mathbf{v}\cdot\varphi)_{H_2(T)} + (\mathbf{f}\cdot\varphi)_{H_2(T)} \tag{3.3}$$

The following iteration process is constructed

$$\mathbf{v}_n = G_T\mathbf{v}_{n-1} + \mathbf{f}$$

where $\mathbf{v}_0$ is an arbitrary vector function, $\mathbf{v}_0 \in H_2(T)$. We call the corresponding process for (1.3) the method of "elastic solutions". By using the Hölder inequality, the characteristic term in the expression $(G_T\mathbf{v}\cdot\varphi)_{H_2(T)}$ is estimated as follows:

$$\left|\int_0^T\int_\Omega\int_0^\tau E(\tau-\alpha)\,\varepsilon(\alpha)\,d\alpha\,\varepsilon_1(\tau)\,d\Omega\,d\tau\right| \leqslant$$

$$m_1\left\{\int_0^T t\int_\Omega\int_0^t \varepsilon^2(\alpha)\,d\alpha\,d\Omega\,dt\right\}^{1/2}\|\varepsilon_1\|_{L_2(\Omega\times[0,\,T])}$$

where

$$m_1 = \max|E(t)| \quad \text{for} \quad t \in [0,\,T].$$

It must be noted that T is arbitrary in this inequality. Summing the estimates for all the members of $(G_T\mathbf{v}\cdot\varphi)_{H_2(T)}$, the estimate

$$\|G_t\mathbf{v}\|_{H_2(t)} \leqslant m_2\left\{\int_0^t \tau\|\mathbf{v}\|^2_{H_2(\tau)}\,d\tau\right\}^{1/2} \tag{3.4}$$

can be obtained by selecting $\varphi = G_T\mathbf{v}$ and using Lemma 2.1, where the constant m_2 depends only on t and is finite for $0 \leqslant t \leqslant T < \infty$. The estimate

$$\|G_T^{k+1}\mathbf{v}\|_{H_2(T)} \leqslant \|\mathbf{v}\|_{H_2(T)}(2^{-1/2}m_2T)^k(k!)^{-1} \tag{3.5}$$

is deduced from (3.4), from which there results that for sufficiently large k the operator $G_T{}^k$ is a compression operator. According to the principle of compressed mappings (3.3) is uniquely solvable. By using the estimate (3.5) the rate of convergence of the series $\mathbf{v}_0 + (\mathbf{v}_1 - \mathbf{v}_0) + (\mathbf{v}_2 - \mathbf{v}_1) + \ldots$ to (3.3) is found. Therefore, the following theorem is valid, from which the first part of the assertion in Theorem 3.1 follows.

Theorem 3.2. Let all the conditions of Theorem 3.1 be satisfied. In this case the sequence of approximate solutions of the method of elastic solutions reduces to a single generalized solution of the linear viscoelasticity problem $\mathbf{u}$ with the velocity

$$\|\mathbf{u}-\mathbf{u}_m\|_{B_1^{n,\,n}(T)} \leqslant K\sum_{k=m}^{\infty}\frac{(MT)^k}{k!}$$

Here the constant M depends only on T, and the constant K on the magnitudes of the corresponding norms of the external forces, the vector function Φ, the initial approximation $\mathbf{u}_0 \in B_1^{n,n}(T)$, constants from the imbedding theorem; M and K are independent of m.

The system of rth approximation equations of the Bubnov-Galerkin method in the terminology of (3.2) is

$$(\mathbf{v}_r^* \cdot \psi_k)_{H_1} = R_1(\mathbf{v}_r^*, \psi_k) + R_2(\Phi, \mathbf{F}, \mathbf{P}, \psi_k), \quad k = 1, \ldots, r$$

The solvability of this integro-differential system for any r is proved just as the solvability of (3.2). The estimate of the norm of the solution $\mathbf{v}_r^*$ obtained is hence independent of the number r. Hence, a weakly convergent subsequence in $H_2(T)$ can be selected from the sequence $\mathbf{v}_r^*$. It is easy to show that this weak limit is indeed the generalized solution of the problem. By virtue of the uniqueness of the solution of the problem the whole sequence converges weakly. Strong convergence of the sequence $\mathbf{v}_r^*$ in the small time segment $[0, T_1]$ follows from the fact that G_t is a compression operator for sufficiently small t. Now, examining the segment $[T_1, 2T_1]$ and taking account of the properties of the operator G_T and the strong convergence of $\mathbf{v}_r^*$ in the space $H_2(T_1)$, it can be seen that the sequence $\mathbf{v}_r^*$ converges strongly in the norm $H_2(2T_1)$, etc., until the whole segment $[0, T]$ is exhausted in a finite number of steps.

Note 1. Let $\Phi \in C^n[0, T; H_1]$ and in the variable φ

$$\int_S \varphi_i C(\partial_t) P_i \, dS + \int_\Omega \varphi_i C(\partial_t) F_i \, d\Omega$$

is a continuous operator in the space $C^\circ[0, T; H_1]$ in the space of continuous functions. In this case the generalized solution of the problem (from Theorem 3) lies in the space $C^n[0, T; H_1]$. This note results from Theorem 1 formulated in [4].

Note 2. In the case of the so-called quasi-static problem (obtained formally from (1.3) for $\rho = 0$) Theorems 3.1, 3.2 are true, where compliance with the condition $m_C + 2 \leqslant m_B$ is optional.

4. Let the vector function be $\varphi \in H_3^\circ(T)$, and for definiteness $m_A = m_B$. Then (1.3) is converted by the following two methods:

$$(\partial_t^n \mathbf{u} \cdot \varphi)_{H_2(T)} - \int_0^T \{(\rho \partial_t^{n+1} \mathbf{u} \cdot \partial_t \varphi)_{L_2(\Omega)} - (\rho c_1 \partial_t^{n+1} \mathbf{u} \cdot \varphi)_{L_2(\Omega)} - \tag{4.1}$$

$$(\rho c_2 \partial_t^n \mathbf{u} \cdot \varphi)_{L_2(\Omega)}\} \, dt = -(\rho \mathbf{a}_{n+1} \cdot \varphi)_{L_2(\Omega)}\big|_{t=0} + \int_0^T R_3(\mathbf{u}, \mathbf{F}, \mathbf{P}, \varphi) \, dt$$

$$\text{for} \quad n = m_B = m_C$$

$$(\partial_t^n \mathbf{u} \cdot \varphi)_{H_2(T)} - \int_0^T \{(\rho \partial_t^n \mathbf{u} \cdot \partial_t \varphi)_{L_2(\Omega)} - (\rho c_1 \partial_t^n \mathbf{u} \cdot \varphi)_{L_2(\Omega)}\} \, dt = \tag{4.2}$$

$$(\rho \mathbf{a}_n \cdot \varphi)_{L_2(\Omega)}\big|_{t=0} + \int_0^T R_4(\mathbf{u}, \mathbf{F}, \mathbf{P}, \varphi) \, dt \qquad \text{for} \quad n = m_B = m_C + 1$$

Here R_3, R_4 are linear forms in each of their variables containing $\partial_t^k \mathbf{u}$ only for $k < n$.

Definition 4.1. The vector function $\mathbf{u} \in B_1^{n, n+1}(T)$ $[\mathbf{u} \in B_3^n(T)]$ satisfying the initial and boundary conditions (1.4), (1.5) and the equalities (4.1), (4.2) for any vector function $\varphi \in H_3^\circ(T)$, where the nth initial condition is taken in the following sense

$$\lim \|\partial_t^n \mathbf{u} - \mathbf{a}_n\|_{L_2(\Omega)} = 0 \quad \text{for} \quad t \to 0$$

is called the generalized solution of the linear viscoelasticity problem in the case $n = m_B = m_C$ $[n = m_B = m_C + 1]$.

Theorem 4.1. Let the following conditions be satisfied:

$$1)\ n = m_B \geqslant m_A,\quad 2)\ \mu > 0,\ \rho > 0,\quad 3)\ \lambda + 2\mu > 0,\ \text{ if } m_B = m_A,$$

$$4)\ \Phi \in B_1^{n+l,\,n+l+1}(T),\ \text{ if } n = m_C,\quad 5)\ \Phi \in B_1^{n+l,\,n+l}(T),\ \text{ if } n = m_C + 1,$$

$$6)\ \partial_t^{k-1} C(\partial_t) F_i \in L_2(\Omega \times [0, T]),\quad 7)\ \partial_t^{k} C(\partial_t) P_i \in L_2(S \times [0, T]),$$

$$8)\ k = 0, \ldots, l,\quad l \geqslant 1.$$

In the case $n = m_C$ $[n = m_C + 1]$: 1) there exists a single generalized solution of the linear viscoelasticity problem in the sense of the definition 4.1 ; 2) the solution can be found by the Bubnov-Galerkin method, where the system of equations of the Bubnov-Galerkin method is uniquely solvable at each step, the sequence of approximate solutions of the Bubnov-Galerkin method lies in some sphere of the space $B_1^{n+l-1,\,n+l}(T)$ $[B_3^{n+l-1}(T)]$ and converges weakly to the generalized solution of the problem $\mathbf{u} \in B_1^{n+l-1,\,n+l}(T)$ $[\mathbf{u} \in B_3^{n+l-1}(T)]$; 3) if $l \geqslant 2$, the sequence $\mathbf{u}_r$ then converges strongly in the space $B_1^{n+l-2,\,n+l-1}(T)$ $[B_3^{n+l-2}(T)]$.

The proof of the theorem of the existence of the Bubnov-Galerkin method for non-stationary problems was first proposed in [5].

The course of the proof in the case $n = m_C$ is described briefly below. As in Sect. 3, the solution is sought in the form

$$\mathbf{u} = \mathbf{u}_0 + \Phi,\qquad \mathbf{u}_0 = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\,\mathbf{v}\,d\tau$$

which reduces to an equivalent problem whose unique solvability involves the existence of a single generalized solution of the linear viscoelasticity problem.

Equivalent problem. Find the vector function $\mathbf{v} \in H_3(T)$ satisfying the following equality

$$(\mathbf{v}\cdot\varphi)_{H_2(T)} - \int_0^T \{(\rho\partial_t\mathbf{v}\cdot\partial_t\varphi)_{L_2(\Omega)} - (\rho c_1\partial_t\mathbf{v}\cdot\varphi)_{L_2(\Omega)} -$$

$$(\rho c_2\mathbf{v}\cdot\varphi)_{L_2(\Omega)}\}\,dt = \int_0^T \left\{R_3\left(\int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\,\mathbf{v}\,d\tau,\ \mathbf{F},\ \mathbf{P},\varphi\right) + R_5(\Phi,\varphi)\right\}dt \tag{4.3}$$

for any vector function $\varphi \in \overset{\circ}{H}_3(T)$ and $\mathbf{v} = 0$ for $t = 0$.

The proof of the existence of the solution of the equivalent problem is by the Bubnov-Galerkin method; the rth approximation of the method

$$\mathbf{v}_r = \sum_{k=1}^{r} \alpha_{rk}(t)\,\psi_k$$

is found from the system of equations

$$\alpha_{rk}''(t) + c_1\alpha_{rk}'(t) + c_2\alpha_{rk}(t) + (\mathbf{v}_r\cdot\psi_k)_{H_1} =$$

$$R_3\left(\int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\,\mathbf{v}_r\,d\tau,\ \mathbf{F},\mathbf{P},\ \psi_k\right) + R_5(\Phi,\psi_k) \tag{4.4}$$

with the initial conditions $\alpha_{rk}(0) = \alpha_{rk}'(0) = 0$, $k = 1,\ldots, r$. Here the ψ_k are elements of the basis of the space H_1 such that

$$\rho\,(\psi_k\cdot\psi_n)_{L_2(\Omega)} = \delta_{kn}$$

The uniqueness of solution of the system of Bubnov-Galerkin Eqs.(4.4) follows from the fact that a system of ordinary differential equations of normal type with constant coefficients is obtained for the corresponding coefficients α_{rk} of the vector function

$$\mathbf{u}_{0r} = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} \mathbf{v}_r \, d\tau$$

The a priori estimate of the solution (4.4) is proved in the space $H_3(T)$, for which the kth equation of (4.4) is multiplied by α_{rk}' and these equalities are then added and integrated with respect to the time between the limits 0 and t. A vector function is contained in the right sides of (4.4) only under the time integral, hence integrating by parts with respect to t in the equality obtained, we have

$$\left| \int_0^t R_3(\mathbf{u}_{0r}, \mathbf{F}, \mathbf{P}, \partial_\tau \mathbf{v}_r) \, d\tau \right| \leqslant M_1 \int_0^t \| \mathbf{v}_r \|_{H_1}^2 d\tau + M_2 \| \partial_\tau \mathbf{v}_r \|_{L_2(\Omega \times [0, T])} + 0.5 \| \mathbf{v}_r \|_{H_1}^2$$

The left side of the equality obtained is of the form

$$\| \partial_t \mathbf{v}_r \|_{L_2(\Omega)}^2 + c_1 \int_0^t \| \partial_\tau \mathbf{v}_r \|_{L_2(\Omega)}^2 d\tau + c_2 \int_0^t (\mathbf{v}_r \cdot \partial_\tau \mathbf{v}_r)_{L_2(\Omega)} d\tau + \| \mathbf{v}_r \|_{H_1}^2$$

A uniform estimate for all $0 \leqslant t \leqslant T$, independent of r

$$\| \mathbf{v}_r \|_{H_1}^2 + \| \partial_t \mathbf{v}_r \|_{L_2(\Omega)}^2 \leqslant M_3 \tag{4.5}$$

results from a comparison between the right and the left sides of the equality and the Gronwall inequality. Here the constants M_k, $k = 1,2,3$ depend only on the magnitudes of the corresponding norms $\mathbf{\Phi}$, $\mathbf{P}$, $\mathbf{F}$ and the length of the segment $[0, T]$. By virtue of the a priori estimate (4.5), a subsequence converging weakly to $\mathbf{v}_0$ in the space $H_3(T)$ can be selected from the sequence of approximations $\mathbf{v}_r$. The imbedding operator $H_3(T)$ in the space $L_2(\Omega)$ is completely continuous, hence $\mathbf{v}_0 = 0$ for $t = 0$.

The system (4.4) reduces to the form (4.3) for which the kth equality in (4.4) is multiplied by $d_k(t)$, the equalities are then added, integrated with respect to time, and where necessary integrated by parts with respect to time. A passage to the limit shows that $\mathbf{v}_0$ satisfies (4.3) for any function φ of the form

$$\varphi = \sum_{k=0}^{N} d_k(t) \psi_k, \qquad d_k(T) = 0 \tag{4.6}$$

Because the set of functions of the form (4.6) is compact in $H_3^\circ(T)$, there results that $\mathbf{v}_0$ is a solution of the equivalent problem. To prove the uniqueness of the solution, we set

$$\varphi = \chi(t) = \int_t^s \mathbf{v}(\tau) \, d\tau, \qquad \chi(t) = 0, \quad \text{if} \quad t \geqslant s$$

in (4.3) for $\mathbf{F} = 0$, $\mathbf{P} = 0$, $\mathbf{\Phi} = 0$. After elementary estimates similar to those made to obtain the inequality (4.5), we obtain

$$\| \chi(0) \|_{H_1}^2 + \| \mathbf{v}(s) \|_{L_2(\Omega)}^2 \leqslant M_4 \left\{ \int_0^s \| \chi(t) \|_{H_3(t)}^2 dt + \| \chi(0) \|_{L_2(\Omega)}^2 \right\}$$

from which the uniqueness of the solution of the equivalent problem results. It follows from the uniqueness of the solution that the whole sequence of approximations $\mathbf{v}_r$ of the Bubnov-Galerkin method converges weakly in $H_3(T)$.

The reasoning relative to Eq. (4.4) differentiated $l-1$ times with respect to time is analogous to that presented above, and terminates the proof of the second part of Theorem 4.1. The strong convergence of the subsequence of approximate solutions of the Bubnov-Galerkin method can be shown by proceeding from the density of a function of the form

$$\sum_{k=0}^{N} c_k(t)\psi_k, \qquad c_k(t) \in C^\infty(0, T)$$

in the space $B_1^{n,m}(T) \cap B_2^n(T)$. The case $n = m_C + 1$ is examined analogously: the problem is reduced to the equivalent problem by the substitution (3.1). A certain difference in the investigation of the equivalent problem from that carried out above for $n = m_C$ is the following: firstly, to obtain the a priori estimate of solutions of the Bubnov-Galerkin system it is necessary to multiply the kth equation of the system by α_{rk}, but not by α_{rk}', and secondly, it is necessary to prove boundedness in the norm $L_2(\Omega)$ for all $0 \leqslant t \leqslant T$ for the limit element of the Bubnov-Galerkin sequence. The method of investigating the equivalent problem agrees completely with the corresponding method of investigating a linear parabolic equation [6].

Note 1. Conditions (6),(7) of Theorem 1 can be replaced by the following

$$\partial_t^k C(\partial_t)\mathbf{F} \in L_2[0, T; H_1^-]$$
$$\partial_t^k C(\partial_t) P_i \in L_2[0, T; W_2^{-1/2}(S)]$$

Note 2. All the results obtained above are valid for the plane viscoelasticity problem.

Note 3. All the results obtained above can be extended to the case of an anisotropic body, where the coefficients of the appropriate polynomials can depend piecewise-continuously on the coordinates x_i and time t.

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BIBLIOGRAPHY

1. Gel'fand, I. M. and Shilov, G. E., Some Questions of the Theory of Differential Equations. Fizmatgiz, Moscow, 1958.
2. Lyons, J.-L. and Magenes, E., Inhomogeneous Boundary Value Problems and Their Application. "Mir", Moscow, 1971.
3. Slobodetskii, L. N., Generalized S.L.Sobolev spaces and their application to the boundary value problems for partial differential equations. Uchen. Zap. A.I. Gertsen, Leningrad Pedag. Inst., Vol.197, 1958.
4. Ciobanu, G. H., On the existence and uniqueness of the solution in linear viscoelasticity. First boundary value problem. Arch. of Mech. Vol.23, №1, 1971.
5. Vorovich, I. I., On the Bubnov-Galerkin method in nonlinear theory of shallow shell vibrations. Dokl. Akad. Nauk SSSR, Vol.110, №5, 1956.
6. Ladyzhenskaia, O. A., Solonnikov, A. A. and Ural'tseva, N. N., Linear and Quasilinear Equations of Parabolic Type. "Nauka", Moscow, 1967.

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